RCMOS – Multidisciplinary Scientific Journal of Knowledge.
ISSN: 2675-9128. São Paulo-SP.

Year V, v.1, n.1, Jan/July 2025. | submission: 2025-12-30 | accepted: 2025-01-01 | published: 2025-01-03

THE IMPACT OF THE METHOD OF SOLVING THIRD-DEGREE EQUATIONS ON THE HISTORY OF ALGEBRA

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SUMMARY

The objective of this article is to study the development of methods for solving third-degree equations from the peoples of Antiquity until the discovery of the algebraic method of resolution that was consolidated during the Renaissance period. Each attempt at resolution was essential, as it enabled advances in the development of Algebra. In addition, the consequences and new discoveries that this method brought to the history of mathematics will be addressed, such as the emergence of complex numbers and the development of Galois Theory. Finally, this study will discuss the feasibility of applying the method in high school, evaluating its pedagogical potential and limitations.

Keywords: Quadratic equation. Renaissance mathematicians. History of algebra

ABSTRACT

The objective of this article is to study the development of methods for solving third-degree equations, from ancient civilizations to the discovery of the algebraic resolution method consolidated during the Renaissance period. Each attempt at resolution was essential, as it enabled advancements in the development of Algebra. Additionally, this study will address the consequences and new discoveries brought by this method to the history of mathematics, such as the emergence of complex numbers and the development of Galois Theory. Finally, this study will discuss the feasibility of applying this method in high school education, evaluating its potential and pedagogical limitations.

Keywords:third-degree equation. Renaissance mathematicians. Historical algebra

1. INTRODUCTION

When a student learns to solve a quadratic equation using the quadratic formula, there is a question as to whether there are formulas for solving higher-degree equations. This question goes back centuries of mathematical development and serves as a gateway to connecting elementary algebra with its historical evolution.

In high school textbooks, the topic of algebraic equations of degree "n" is mainly addressed in the study of polynomial equations of one variable based on the "Fundamental Theorem of Algebra". However, such an approach is insufficient as it leaves significant gaps in the resolution of algebraic equations and may discourage curious students from finding connections between elementary Algebra and historical and theoretical advances in Mathematics.

When faced with a third-degree equation, for example, the student will question whether there is a formula to solve it. The study of solving third-degree equations, in addition to providing a rich mathematical content to be explored, has great historical importance, as it developed at a time when knowledge and research were at their peak, including intellectual disputes that generated important discoveries in the field of Algebra. In addition, there are important personalities involved in this process, such as Scipione del Ferro, Tartaglia and Girolamo Cardano.

This article seeks to explore this fascinating history, as well as explain the methods developed, their historical implications and the consequences of this fact for the development of modern Algebra.

2. HISTORICAL CONTEXTUALIZATION

Since ancient times, the art of solving equations has been present in the development of Mathematics. The beginning of solving third-degree equations occurred with the Babylonians,

who constructed tables of squares and cubes of numbers to help solve a third-degree equation.

Geometric problems related to the volume of solids challenged Greek mathematicians to solve equations unknown at the time. The main problem that contributed to the development of third-degree equations in Antiquity, both for the Greeks and the Egyptians, was the duplication of a cube, the challenge of which was to obtain a cube with twice the volume from an existing cube.

Arab mathematicians inherited a vast mathematical knowledge from the Greeks and contributed to the study of solving equations. During this period, the mathematician Omar Khayyam stood out, when he began a study to determine the solution of a third-degree equation geometrically, through the intersection of conic sections, since he considered it impossible to solve it through an algebraic method. This argument motivated mathematicians to explore new approaches to solving such problems.

In the Middle Ages, the famous mathematician Leonardo Fibonacci worked indirectly on solving algebraic equations, including quadratic equations, with an approach involving practical and geometric methods, such as finding the solution with a ruler and compass. Such methods were often applied to commercial and financial problems. Fibonacci explored applied and practical solutions by finding approximate solutions to equations, and his ideas, in some ways, coincide with the thinking of Omar Khayyam who claimed that it was impossible to find the solution of a quadratic equation by algebraic methods. Although Khayyam considered an algebraic solution impossible, his geometric ideas paved the way for approaches that would be taken up by European mathematicians centuries later.

During the Renaissance, the study of classical mathematics was revived, linked to the desire to solve practical problems. During this period, quadratic equations received notable attention, with the leading role of three Italian mathematicians: Scipione del Ferro, Tartaglia and Cardano. Each of them made significant contributions to the solution of quadratic equations amidst disputes, fights and betrayals, which left an important legacy in the history of mathematics, boosting Algebra as a discipline. These mathematicians fought true academic battles to find solutions to a problem that had intrigued humanity for centuries. In the 16th century, the world learned about Cardano's method for solving a quadratic equation with the publication of his work.*Ars Magna*, considered a milestone in the history of Algebra.

3. CONTRIBUTIONS OF RENAISSANCE MATHEMATICIANS TO THE SOLVING OF THIRD-DEGREE EQUATIONS

For centuries, finding a systematic solution to a third-degree equation seemed an impossible task. These equations frequently appeared in geometric and financial problems, but a method for solving them was out of reach, which puzzled mathematicians. In this context, the Italian mathematics professor Scipione del Ferro is considered the pioneer in solving third-degree equations of the form , whose coefficients are positive numbers. He discovered an algebraic method that solved such equations, preferring to keep this discovery secret, revealing it only to his students, among whom was Antonio Maria Fiore.

Niccolo Fontana, better known as Tartaglia, a self-taught mathematician, revealed that he could also solve quadratic equations. Fiore challenged him to a duel, a common practice at the time when scholars would confront each other by proposing problems to each other. Fiore proposed only equations of the form , as he trusted the method of solving taught by Del Ferro. However, Tartaglia had already mastered this technique, as well as already knew the method for equations of the form , which Fiore was unaware of. Fiore was unable to solve any of the problems proposed by Tartaglia, which culminated in his defeat.

With his victory in the duel, Tartaglia gained notoriety, attracting the attention of mathematician and physician Girolamo Cardano, who was fascinated by Tartaglia's skill. Cardano's goal was to understand how Tartaglia managed to solve the third-degree equations, promising to keep it secret if he revealed the long-sought method of resolution. Tartaglia was reluctant to share his method of resolution, but was later convinced by Cardano. Tartaglia revealed it through the following poem, whose verses in Italian describe the main steps of his method.

When the cube comes together, if
aqquaglia is a discrete number, I'll do
a duo altri differenti in that

Then I'll have a question for consueto
Che"llor productto semper sia equale
Alterzo delle cube cose grandson,
The residue is generally there and
then sweeps away your main thing. In
the second part of cotestiatti When
the cube remains alone You
osseruarai questaltri contratti, From
the number farai due such a part'`a
uolo Che luna in later itself produces
schietto El third cube cose in stolo
Delle qual poi, per communprecetto
Torrai li lati cubi insieme gionti

Et cotal somma sara il tuo concetto. El
terzo poi de questi nostri conti Se solue
col secondo se ben guardi Che per
natura son quasi congionti. Questi
trouai, non con passi tardi Nel mille
cinquecent`e, foure trenta Con
fondamenti ben sald`e gagliardi Nella
citta dal marintorno centa.

In 1545, Cardano published*Ars Magna*, in which he detailed the solution of third-degree equations. Tartaglia considered this attitude a betrayal of his trust, increasing tension between them. This controversial fact marked one of the first recorded cases of dispute over intellectual property in mathematics.

4. PROOF FOR THE GENERAL CASE OF A THIRD DEGREE EQUATION

Be with the general form of a third-degree equation.

The method consists of eliminating the second-degree term through algebraic operations, and obtaining an equation of the type:

Replacing the variable with in the general form and grouping similar terms, we will have:

Equating the second-degree term to zero, we obtain:

Substituting into the equation:

$$z^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right) z + \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = 0$$

Now we will obtain peq as a function of comparing the equation obtained with the equation:

So, we arrive at:

$$p = \frac{c}{3a} - \frac{b^2}{9a^2}$$

$$q = \frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a}$$

It was already known at the time that a root of equation (I) could be obtained from the roots of the following second-degree equation:

The way to obtain a root of the equation

would be as follows:

Solving the second-degree equation:

$$y_1 = q + \sqrt{q^2 + p^3}$$
$$y_2 = q - \sqrt{q^2 + p^3}$$

Replacing:

$$x_1 = \sqrt[3]{q + \sqrt{q^2 + p^3}} + \sqrt[3]{q - \sqrt{q^2 + p^3}} - \frac{b}{3a}$$

Note that if , we have to manipulate complex numbers. Initially, complex numbers were ignored and only received due importance in later studies.

To calculate the other roots, we simply divide the polynomial by . Thus, we will fall back into an equationdogof second degree, which we can solve. Let the equation that was obtained from the divisionto thefrom the general equation by . Solving the equation , we obtain

$$x_2 = \frac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$$

$$x_3 = \frac{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$$

The roots are in function of We want to express them in function of . To do this, we use the Briot-Ruffini device:

From this we conclude:

The rest of the divisionandgiven by:

.

Therefore,

We substitute to express:

$$c_1 = \frac{-d}{x_1}.$$

Thus, the Briot-Ruffini device allows us to simplify the expressions of the roots in terms of the coefficients and the known root . Substituting in x2 and x3, we obtain:

$$x_2 = \frac{-(ax_1 + b) + \sqrt{(ax_1 + b)^2 - 4a \cdot \left(-\frac{d}{x_1}\right)}}{2a},$$

$$x_3 = \frac{-(ax_1 + b) - \sqrt{(ax_1 + b)^2 - 4a \cdot \left(-\frac{d}{x_1}\right)}}{2a}$$

Note that if

5.APPLYDOGFROM THE CARDANO-TARTAGLIA METHOD

Consider the equation given by

We will apply the method to find its roots

$$p = \frac{c}{3a} - \frac{b^2}{9a^2} = \frac{6}{3} - \frac{(-6)^2}{9} = 2 - 4 = -2$$

$$q = \frac{bc}{6a^2} - \frac{d}{2a} - \frac{b^3}{27a^3} = \frac{(-6)(6)}{6} - \frac{-5}{2} - \frac{(-6)^3}{27}$$

$$q = -6 + 8 + 5 = \frac{7}{2}$$

$$q^2 + p^3 = \left(\frac{q}{2}\right)^2 = \left(\frac{9}{2}\right)^2 = \frac{81}{4} + 8 = \frac{49}{4}$$

$$x_1 = \sqrt[3]{q + \sqrt{q^2 + p^3}} + \sqrt[3]{q - \sqrt{q^2 + p^3}} - \frac{b}{3a}$$

$$x_1 = \sqrt[3]{\frac{9}{2} + \sqrt{\frac{49}{4}}} + \sqrt[3]{\frac{9}{2} - \sqrt{\frac{49}{4}}} + 2$$

$$x_1 = \sqrt[3]{8} + \sqrt[3]{1} + 2 = 2 + 1 + 2 = 5$$

Therefore, . We will obtain and

$$x_2 = \frac{-(ax_1+b)+\sqrt{(ax_1+b)^2-4a\left(-\frac{d}{x_1}\right)}}{2a}$$

$$x_2 = \frac{-(1(5)-6)+\sqrt{((1)(5)+(-6))^2-4(1)\left(-\frac{(-5)}{5}\right)}}{2}$$

$$x_2 = \frac{1+\sqrt{1-4(1)(1)}}{2} = \frac{1+\sqrt{-3}}{2} = \frac{1+\sqrt{3}\,i}{2}$$

Similarly,

$$x_3 = \frac{1-\sqrt{3}\,i}{2}$$

The roots of the equation are:

$$x_1 = 5, \quad x_2 = \frac{1+\sqrt{3}\,i}{2} \quad x_3 = \frac{1-\sqrt{3}\,i}{2}$$

6. CONSEQUENCES AFTER DISCOVERING THE RESOLUTION METHOD

The publication of*Ars Magna*, despite being the first work to present an algebraic method for third-degree equations, generated controversy between Cardano and Tartaglia. Tartaglia was deeply outraged, as he claimed that Cardano broke his promise to keep the resolution method secret. This dispute was one of the most well-known controversies in the history of Mathematics, showing that questions of ethics and ego were already present in the scientific community.

According to Lima (1999),

We should note, however, that Cardano was the first mathematician to manipulate complex numbers as if they were any numbers, solving a third-degree equation using the method described in Ars Magna. When, at the end of the solution, he found numbers of the form a + b√−1, Cardano classified them as "useless". Bombelli, however, not only manipulated such strange entities, but also presented laws of multiplication, division and addition. (LIMA, 1999, p. 18).

Furthermore, Ludovico Ferrari, a disciple of Cardano, developed a method for solving fourth-degree equations. These discoveries served as the basis for more sophisticated concepts such as Galois theory, developed centuries later, which proved that equations of degree higher than fourth cannot be solved by similar algebraic methods.

7. THE FEASIBILITY OF THE CARDANO-TARTAGLIA METHOD IN SECONDARY EDUCATION

Integrating the Cardano-Tartaglia Method into secondary education would enrich the study of Algebra and its history. history, showing that mathematical concepts result from efforts from different eras. However, their complexity can make it difficult to include them in the regular curriculum. Their use in interdisciplinary projects or classes for advanced classes would be more viable, promoting historical and mathematical thinking.

On the other hand, there are limitations to including this topic in the high school curriculum. First, the method requires extensive knowledge in the area of Algebra, such as the manipulation of radicals and complex numbers, which are abstract concepts for students. In addition, it would make the curriculum denser, requiring greater content planning so that students do not feel overwhelmed by concepts that are difficult to assimilate. Finally, this content requires adequate training for teachers so that they do not have difficulty explaining it in the classroom, since it is a complex topic.

Despite the challenges, it is possible to integrate the Cardano-Tartaglia Method into the high school curriculum in specific concepts, such as interdisciplinary projects, extracurricular activities or classes on the topic for advanced classes. The combination of the topic with the history of mathematics promotes a connection between students and the historical context, awakening their interest in the importance of algebraic methods in the development of mathematical thought throughout the centuries.

8. FINAL CONSIDERATIONS

The development of a method for solving third-degree equations, from Antiquity to the Renaissance, generated a revolution in the field of Algebra, such as the emergence of complex numbers and an in-depth study of methods for solving higher-degree equations.

Despite the existence of the Cardano-Tartaglia formula for third-degree equations, it is clear that it does not have the same simplicity as the quadratic formula, since its applicability requires a great deal of algebraic knowledge. This makes its approach in the classroom in high school challenging; however, it can contribute significantly to the learning of advanced students and highlight the importance of mathematical methods in contributing to scientific progress and the transformative role that mathematics has played in society since the beginning.

Solving third-degree equations revolutionized Algebra and paved the way for discoveries such as complex numbers and Galois Theory. Despite its difficulty, the method demonstrates the importance of Mathematics in the history of scientific thought, connecting the past to the present and contributing to advances in various areas.

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